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ON LAPLACE TYPE PROBLEMS (I)

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Abstract

In this paper, we consider some Laplace type problems for lattices with axial symmetry and different types of obstacles. We compute the probability that a segment of random position and constant length intersects a side of the lattice.

1. Cell with Two Rectangles and Eight Triangles Obstacles

Let $\Re_1(a, b; m)$ be the lattice with the fundamental cell $C_0^{(1)}$ composed of a rectangle $C_{01}^{(1)}$ of sides $4a$ and b and a rectangle $C_{02}^{(1)}$ of sides $2a$ and b and with eight triangles isosceles obstacles of sides $\frac{m}{2}$, $\frac{m}{2}$, $\frac{m\sqrt{2}}{2}$, with $0 \leq m \leq \min(2a, b)$ (Figure 1).

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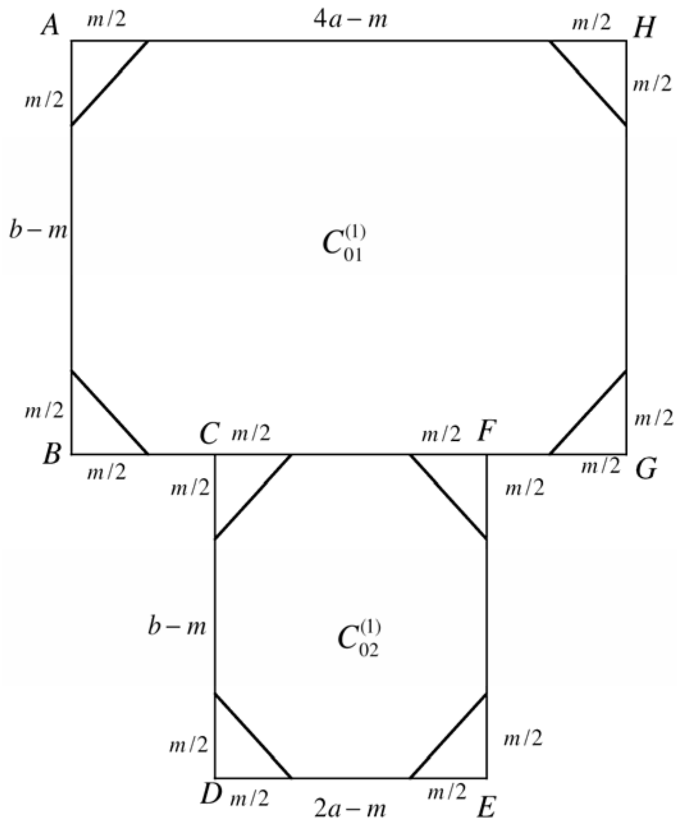

Figure 1.

We have

$$\text{area } C_{01}^{(1)} = 4ab - \frac{m^2}{2},$$

$$\text{area } C_{02}^{(1)} = 2ab - \frac{m^2}{2}.$$

Considering a segment s of random position and of constant length $l < \min(2a - m, b - m)$, called *body test*, we want to compute the probability that the segment intersects a side of the lattice, therefore, the probability $P_{\text{int}}^{(1)}$ that the segment s intersects a side of the fundamental cell $C_0^{(1)}$.

The position of the segment s is determinated by its middle point O and by the angle φ that it forms with the side BG (or DE) in the cell $C_0^{(1)}$.

In order to compute the probability $P_{\text{int}}^{(1)}$, we consider before the limited positions of the segment s, for a fixed value of the angle φ, situated in $C_{01}^{(1)}$ and then the limited positions of the segment s, for the same value of the angle φ, situated in $C_{02}^{(1)}$ (Figure 2).

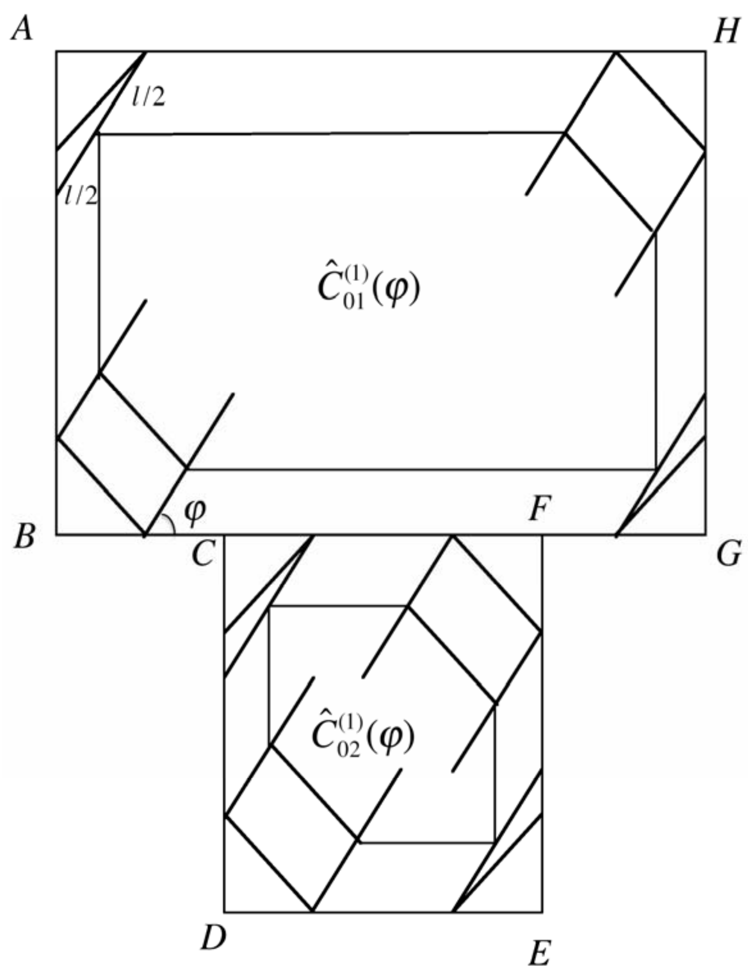

Figure 2.

Denoting with $\widehat{C}_{01}^{(1)}(\varphi)$, the polygon determined from the limited positions of the segment s in the first case and with $\widehat{C}_{02}^{(1)}(\varphi)$ in the second case.

Considering a result obtained in the previous paper [1], we can write

$$\text{area } \widehat{C}_{01}^{(1)}(\varphi) = 4ab - \frac{m^2}{2} - \left[bl\cos\varphi + \left(4a - \frac{m}{2}\right)l\sin\varphi - \frac{l^2}{2}\sin 2\varphi\right],$$

$$\text{area } \widehat{C}_{02}^{(1)}(\varphi) = 2ab - \frac{m^2}{2} - \left[bl\cos\varphi + \left(2a - \frac{m}{2}\right)l\sin\varphi - \frac{l^2}{2}\sin 2\varphi\right]. \tag{1}$$

Denoting with $M_i^{(1)}$, the set of segments s that have the middle point O in $C_{0i}^{(1)}(\varphi)$ and with $N_i^{(1)}$, the set of segments s completely contained in $C_{0i}^{(1)}\ (i = 1, 2)$, we have [3]

$$P_{\text{int}}^{(1)} = 1 - \frac{\mu\left(N_1^{(1)}\right) + \mu\left(N_2^{(1)}\right)}{\mu\left(M_1^{(1)}\right) + \mu\left(M_2^{(1)}\right)}, \tag{2}$$

where μ is the Lebesgue measure in Euclidean plane.

In order to compute the measures $\mu\left(M_i^{(1)}\right)$ and $\mu\left(N_i^{(1)}\right)$, we use the Poincaré kinematic measure [2]

$$dK = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinates of the middle point O of the segment s and φ is the defined angle.

Because $\varphi \in \left[0, \frac{\pi}{2}\right]$, we have

$$\mu\left(M_1^{(1)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\,y)\in C_{01}^{(1)}\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } C_{01}^{(1)}\right) d\varphi = \frac{\pi}{2}\left(4ab - \frac{m^2}{2}\right),$$

$$\mu\left(M_2^{(1)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\,y)\in C_{02}^{(1)}\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } C_{02}^{(1)}\right) d\varphi = \frac{\pi}{2}\left(2ab - \frac{m^2}{2}\right), \tag{3}$$

and considering the relations (1),

$$\mu\left(N_1^{(1)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\,y)\in \widehat{C}_{01}^{(1)}(\varphi)\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } \widehat{C}_{01}^{(1)}(\varphi)\right) d\varphi = \frac{\pi}{2}\left(4ab - \frac{m^2}{2}\right)$$

$$- \left(4a + b - \frac{m}{2}\right) l + \frac{l^2}{2},$$

$$\mu\left(N_2^{(1)}\right) = \frac{\pi}{2}\left(2ab - \frac{m^2}{2}\right)$$

$$- \left(2a + b - \frac{m}{2}\right) l + \frac{l^2}{2}. \tag{4}$$

From the relations (2), (3), and (4) give us

$$P_{\text{int}}^{(1)} = \frac{2(6a + 2b - m)l - 2l^2}{\pi(6ab - m^2)}. \quad (5)$$

For $m = 0$, the obstacles become points and the probability (5) becomes

$$P_{\text{int}}^{(1)} = \frac{2(3a + b)l - l^2}{3\pi ab} = P, \quad (6)$$

therefore, the Laplace probability for a rectangle of sides $3a$ and b that represented the media of the rectangles $C_{01}^{(1)}$ and $C_{02}^{(1)}$.

2. Cell with Three Rectangles and Twelve Triangles Obstacles

Let $\Re_2(a, b; m)$ be the lattice with the fundamental cell $C_0^{(2)}$ composed of a rectangle $C_{01}^{(2)}$ of sides $2a$ and b, a rectangle $C_{02}^{(2)}$ of sides $4a$ and b, and a rectangle $C_{03}^{(2)}$ of sides $6a$ and b and with twelve triangles isosceles obstacles of sides $\frac{m}{2}, \frac{m}{2}, \frac{m\sqrt{2}}{2}$ (Figure 3).

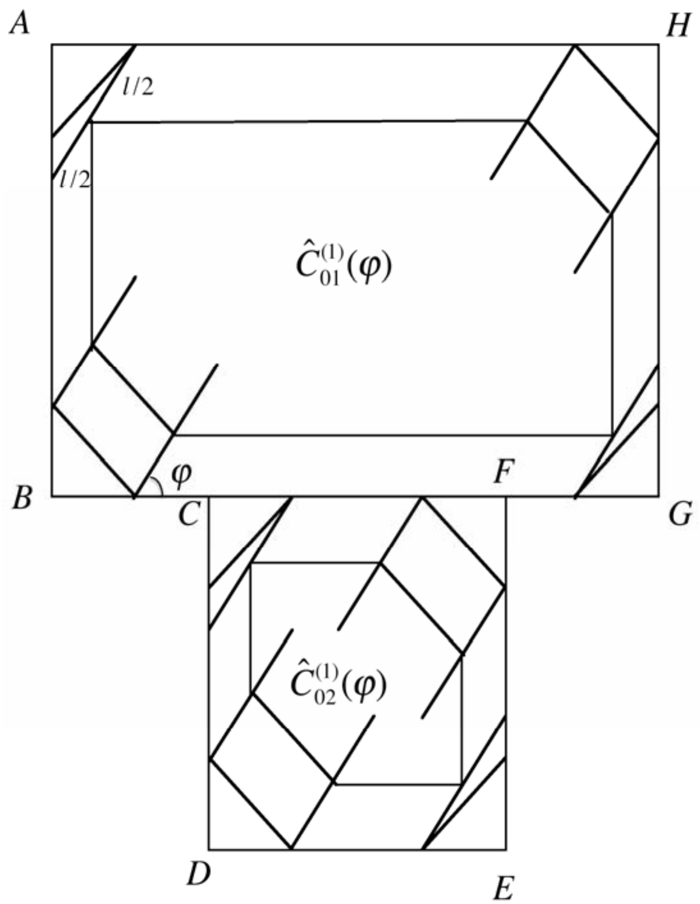

Figure 3.

We have

$$\text{area } C_{01}^{(2)} = 2ab - \frac{m^2}{2},$$

$$\text{area } C_{02}^{(2)} = 4ab - \frac{m^2}{2},$$

$$\text{area } C_{03}^{(2)} = 6ab - \frac{m^2}{2}.$$

Considering a segment s of random position and of constant length $l < \min(2a - m, b - m)$, called *body test*, we want to compute the probability that this segment intersects a side of the lattice, therefore, the probability $P_{\text{int}}^{(2)}$ that the segment s intersects a side of the fundamental cell $C_0^{(2)}$.

The position of the segment s is determinated by its middle point O and by the angle φ that it forms with the side BN in the cell $C_0^{(2)}$.

In order to compute the probability $P_{\text{int}}^{(2)}$, we consider the limited positions of the segment s, for a fixed value of φ, situated in the rectangle $C_{0i}^{(2)} (i = 1, 2, 3)$.

Denoting with $\widehat{C}_{0i}^{(2)}(\varphi) (i = 1, 2, 3)$, the polygon determined from the limited positions of the segment s situated in the rectangle $C_{0i}^{(2)} (i = 1, 2, 3)$.

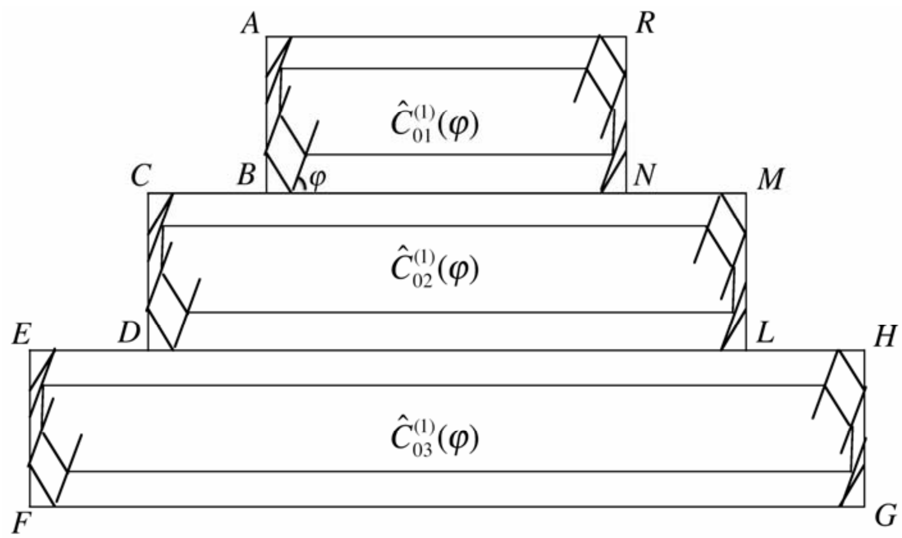

Figure 4.

Considering a result obtained in the previous paper [1], we can write

$$\text{area } \widehat{C}_{01}^{(2)} = 2ab - \frac{m^2}{2}$$

$$-\left[bl\cos\varphi+\left(2a-\frac{m}{2}\right)l\sin\varphi-\frac{l^2}{2}\sin 2\varphi\right],$$

$$\text{area } \widehat{C}_{02}^{(2)} = 4ab-\frac{m^2}{2}$$

$$-\left[bl\cos\varphi+\left(4a-\frac{m}{2}\right)l\sin\varphi-\frac{l^2}{2}\sin 2\varphi\right],$$

$$\text{area } \widehat{C}_{03}^{(2)} = 6ab-\frac{m^2}{2}$$

$$-\left[bl\cos\varphi+\left(6a-\frac{m}{2}\right)l\sin\varphi-\frac{l^2}{2}\sin 2\varphi\right]. \tag{7}$$

Denoting with $M_i^{(2)}$, the set of segments s that have the middle point O in $\widehat{C}_{01}^{(2)}(\varphi)$ and with $N_i^{(2)}$, the set of segments s completely contained in $C_{0i}^{(2)}$ $(i = 1, 2, 3)$, we have [3]

$$P_{\text{int}}^{(2)} = 1-\frac{\mu\left(N_1^{(2)}\right)+\mu\left(N_2^{(2)}\right)+\mu\left(N_3^{(2)}\right)}{\mu\left(M_1^{(2)}\right)+\mu\left(M_2^{(2)}\right)+\mu\left(M_3^{(2)}\right)}, \tag{8}$$

where μ is the Lebesgue measure in Euclidean plane.

In order to compute the measures $\mu\left(M_i^{(2)}\right)$ and $\mu\left(N_i^{(2)}\right)$, we use the Poincaré kinematic measure [2]

$$dK = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinates of the middle point O of the segment s and φ is the defined angle.

We have $\varphi \in \left[0, \frac{\pi}{2}\right]$, therefore;

$$\mu\left(M_1^{(2)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\in C_{01}^{(2)}\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } C_{01}^{(2)}\right) d\varphi = \frac{\pi}{2}\left(2ab - \frac{m^2}{2}\right),$$

$$\mu\left(M_2^{(2)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in C_{02}^{(2)}\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } C_{02}^{(2)}\right) d\varphi = \frac{\pi}{2}\left(4ab - \frac{m^2}{2}\right),$$

$$\mu\left(M_3^{(2)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in C_{03}^{(2)}\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } C_{03}^{(2)}\right) d\varphi = \frac{\pi}{2}\left(6ab - \frac{m^2}{2}\right), \tag{9}$$

and considering the relations (7),

$$\mu\left(N_1^{(2)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in \widehat{C}_{01}^{(2)}(\varphi)\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } \widehat{C}_{01}^{(2)}(\varphi)\right) d\varphi$$

$$= \frac{\pi}{2}\left(2ab - \frac{m^2}{2}\right) - \int_0^{\frac{\pi}{2}} [bl \cos\varphi$$

$$+ \left(2a - \frac{m}{2}\right) l \sin\varphi - \frac{l^2}{2} \sin 2\varphi]$$

$$= \frac{\pi}{2}\left(2ab - \frac{m^2}{2}\right) - \left(2a + b - \frac{m}{2}\right)l + \frac{l^2}{2},$$

$$\mu\left(N_2^{(2)}\right) = \frac{\pi}{2}\left(4ab - \frac{m^2}{2}\right)$$

$$-\left(4a + b - \frac{m}{2}\right)l + \frac{l^2}{2},$$

$$\mu\left(N_3^{(2)}\right) = \frac{\pi}{2}\left(6ab - \frac{m^2}{2}\right)$$

$$-\left(6a + b - \frac{m}{2}\right)l + \frac{l^2}{2}. \tag{10}$$

From the relations (8), (9), and (10) follow

$$P_{\text{int}}^{(2)} = \frac{2\left(12a + 3b - \frac{3m}{2}\right)l - 3l^2}{\pi\left(12ab - \frac{3m^2}{2}\right)}. \tag{11}$$

For $m = 0$, the obstacles become points and the probability (11) is written as

$$P_1 = \frac{2(4a + b) - l^2}{4\pi ab},$$

therefore, the Laplace probability for a rectangle of sides $4a$ and b that represented the media of the rectangles $C_{01}^{(2)}$, $C_{02}^{(2)}$, and $C_{03}^{(2)}$.

3. Cell with Two Rectangles and Eight Circular Sectors Obstacles

Let $\Re_3(a, b; m)$ be the lattice with the fundamental cell $C_0^{(3)}$ composed of a rectangle $C_{01}^{(3)}$ of sides $4a$ and b and a rectangle $C_{02}^{(3)}$ of sides $2a$ and b and with eight obstacles quarters of a circle of radius $\frac{m}{2}$ (Figure 5).

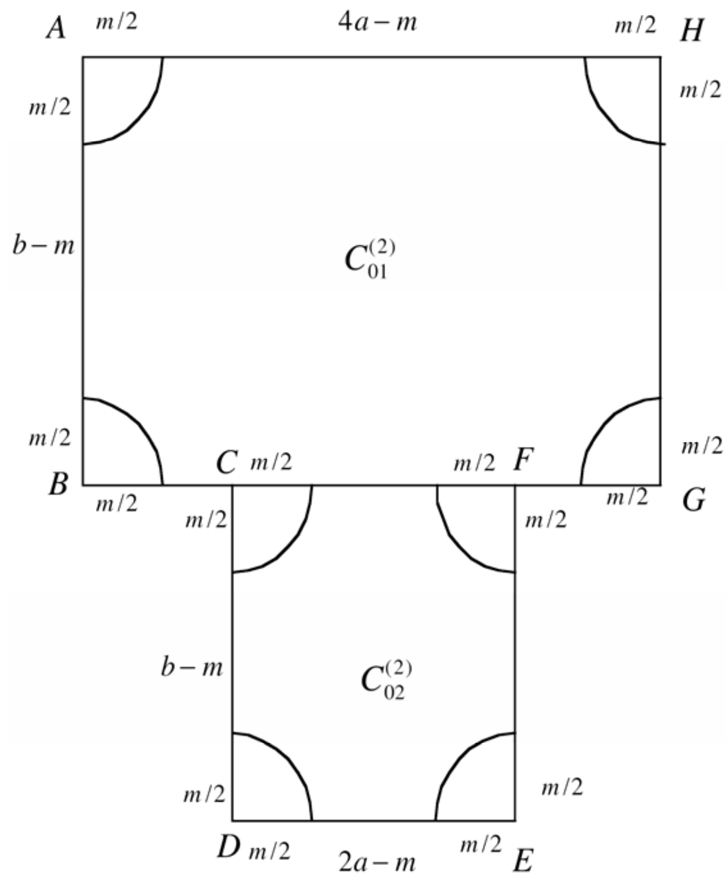

Figure 5.

We have

$$\text{area } C_{01}^{(3)} = 4ab - \frac{\pi m^2}{4},$$

$$\text{area } C_{02}^{(3)} = 2ab - \frac{\pi m^2}{4}.$$

Considering the same body test of the point 1, we want to compute the probability that the segment s intersects a side of the lattice, therefore, the probability $P_{\text{int}}^{(3)}$ that the body test intersects a side of the fundamental cell $C_0^{(3)}$.

In order to compute the probability $P_{\text{int}}^{(3)}$, we consider before the limited positions of the segment s, for a fixed value of the angle φ, situated in $C_{01}^{(3)}$ and then the limited positions of the segment s, for the same value of the angle φ, situated in $C_{02}^{(3)}$ (Figure 6).

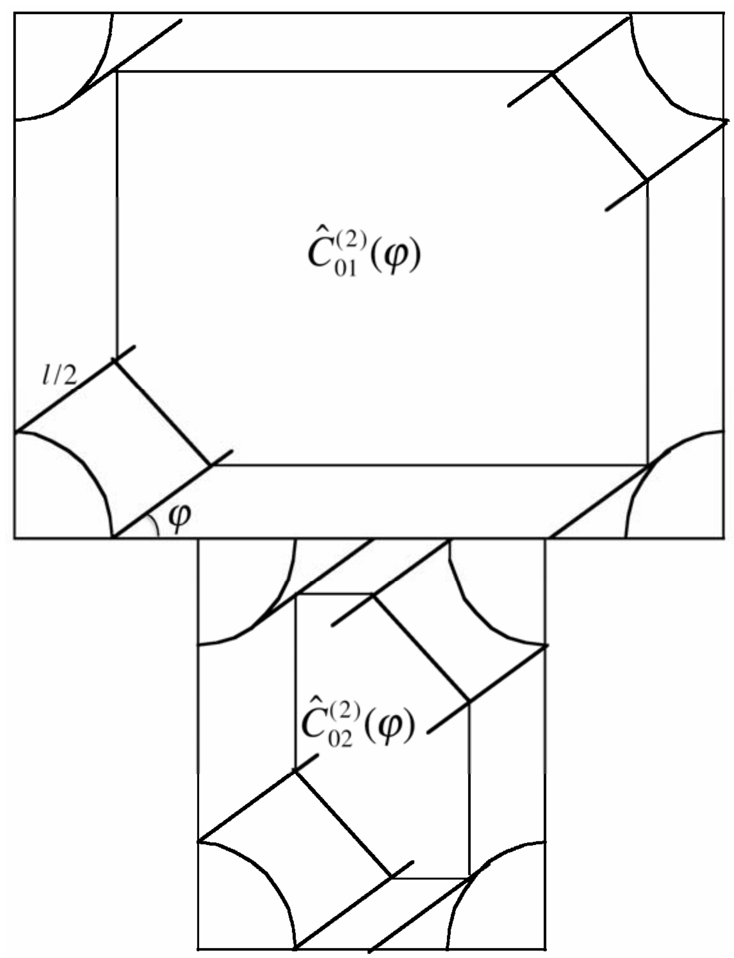

Figure 6.

Let $\widehat{C}_{01}^{(3)}(\varphi)$ be the polygon determined from the limited positions in the first case and $\widehat{C}_{02}^{(3)}(\varphi)$ in the second case.

Considering a result obtained in the previous paper [1], we can write

$$\text{area}\,\widehat{C}_{01}^{(3)} = 4ab - \frac{\pi m^2}{4} - \frac{m^2}{4}(1-\pi)$$

$$- [\, bl\cos\varphi + (4a-m)\, l\sin\varphi - \frac{l^2}{2}\sin 2\varphi \,],$$

$$\text{area}\,\widehat{C}_{02}^{(3)} = 2ab - \frac{\pi m^2}{4} - \frac{m^2}{4}(1-\pi)$$

$$- [\, bl\cos\varphi + (2a-m)\, l\sin\varphi - \frac{l^2}{2}\sin 2\varphi \,]. \tag{12}$$

Denoting with $M_i^{(3)}$, the set of segments s that have the middle point O in $\widehat{C}_{0i}^{(3)}$ and with $N_i^{(3)}$, the set of segments s completely contained in $C_{0i}^{(3)}$ $(i = 1, 2)$, we have [3]

$$P_{\text{int}}^{(3)} = 1 - \frac{\mu\left(N_1^{(3)}\right) + \mu\left(N_2^{(3)}\right)}{\mu\left(M_1^{(3)}\right) + \mu\left(M_2^{(3)}\right)}. \tag{13}$$

With the notations of the point 1, we have

$$\mu\left(M_1^{(3)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\in C_{01}^{(3)}\}} dxdy$$

$$= \frac{\pi}{2}\,\text{area}\,C_{01}^{(3)} = \frac{\pi}{2}\left(4ab - \frac{\pi m^2}{4}\right),$$

$$\mu\left(M_2^{(3)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\in C_{02}^{(3)}\}} dxdy$$

$$= \frac{\pi}{2} \text{ area } C_{02}^{(3)} = \frac{\pi}{2}\left(2ab - \frac{\pi m^2}{4}\right), \tag{14}$$

and considering the relations (12),

$$\mu\left(N_1^{(3)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in \widehat{C}_{01}^{(3)}(\varphi)\}} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left(\text{area } \widehat{C}_{01}^{(3)}(\varphi)\right) d\varphi$$

$$= \frac{\pi}{2}\left(4ab - \frac{\pi m^2}{4}\right) - \frac{\pi(1-\pi)\, m^2}{8}$$

$$- \int_0^{\frac{\pi}{2}} \left[\, bl \cos\varphi + (4a - m)\, l \sin\varphi - \frac{l^2}{2} \sin 2\varphi \,\right] d\varphi$$

$$= \frac{\pi}{2}\left(4ab - \frac{\pi m^2}{4}\right) - \frac{\pi(1-\pi)\, m^2}{8} - \left[(4a + b - m)\, l - \frac{l^2}{2}\right],$$

$$\mu\left(N_2^{(3)}\right) = \frac{\pi}{2}\left(2ab - \frac{\pi m^2}{2}\right) - \frac{\pi(1-\pi)\, m^2}{8}$$

$$- \left[(2a + b - m)\, l - \frac{l^2}{2}\right]. \tag{15}$$

The relations (13), (14), and (15) give us

$$P_{\text{int}}^{(3)} = \frac{\frac{\pi(1-\pi)m^2}{2} + \left[4(3a + b - m)l - 2l^2\right]}{\pi\left(6ab - \frac{\pi m^2}{2}\right)}. \tag{16}$$

For $m = 0$, the obstacles become points and the probability (16) becomes

$$P_{\text{int}}^{(3)} = \frac{2(3a + b)l - l^2}{3\pi ab} = P.$$

4. Cell with Three Rectangles and Twelve Circular Sectors Obstacles

Let $\Re_4(a, b; m)$ be the lattice with fundamental cell $C_0^{(4)}$ composed of a rectangle $C_{01}^{(4)}$ of sides $2a$ and b, a rectangle $C_{02}^{(4)}$ of sides $4a$ and b and a rectangle $C_{03}^{(4)}$ of sides $6a$ and b and with twelve obstacles quarters of a circle of radius $\frac{m}{2}$ (Figure 7).

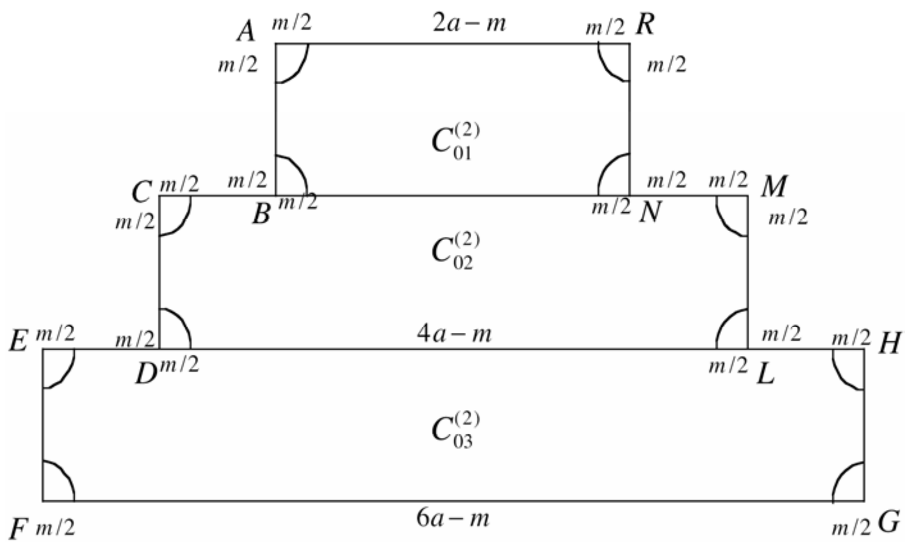

Figure 7.

We have

$$\text{area } C_{01}^{(4)} = 2ab - \frac{m^2}{4},$$

$$\text{area } C_{02}^{(4)} = 4ab - \frac{m^2}{4},$$

$$\text{area } C_{03}^{(4)} = 6ab - \frac{m^2}{4}.$$

Considering the same body test s of the point 1, we want to compute the probability $P_{\text{int}}^{(4)}$ that the segment s intersects a side of the fundamental cell $C_0^{(4)}$.

In order to compute this probability, we consider the limited positions of the segment s, for a fixed value of φ, situated in the rectangle $C_{0i}^{(4)}$ $(i = 1, 2, 3)$, and let $\widehat{C}_{0i}^{(4)}(\varphi)$ be the polygon determined from these positions (Figure 8).

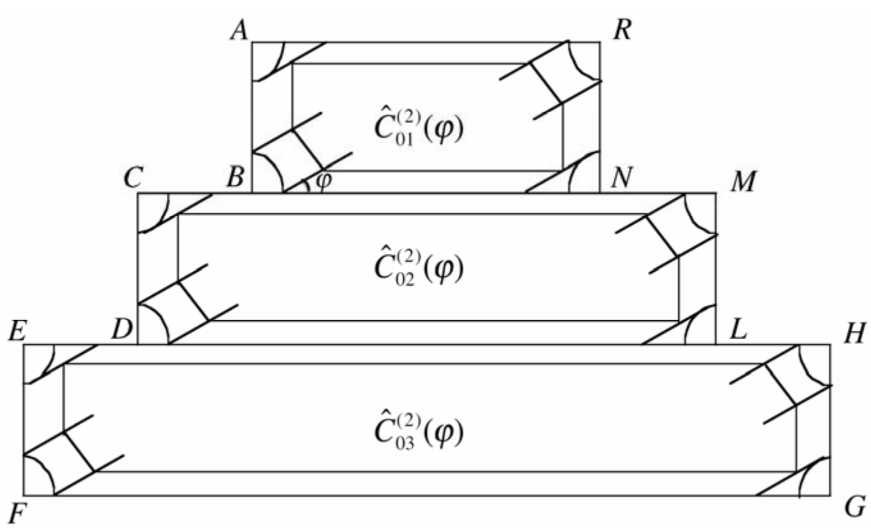

Figure 8.

In the previous paper [1] give us

$$\text{area } \widehat{C}_{01}^{(4)} = 2ab - \frac{m^2}{4}$$

$$- [\, bl \cos\varphi + (2a - m)\, l \sin\varphi - \frac{l^2}{2} \sin 2\varphi \,],$$

$$\text{area } \widehat{C}_{02}^{(4)} = 4ab - \frac{m^2}{4}$$

$$-[\, bl \cos\varphi + (4a - m)\, l \sin\varphi - \frac{l^2}{2} \sin 2\varphi \,],$$

$$\text{area } \widehat{C}_{03}^{(4)} = 6ab - \frac{m^2}{4}$$

$$-[\, bl \cos\varphi + (6a - m)\, l \sin\varphi - \frac{l^2}{2} \sin 2\varphi \,]. \tag{17}$$

Similarly to the formula (2), we can write

$$P_{\text{int}}^{(4)} = 1 - \frac{\mu\left(N_1^{(4)}\right) + \mu\left(N_2^{(4)}\right) + \mu\left(N_3^{(4)}\right)}{\mu\left(M_1^{(4)}\right) + \mu\left(M_2^{(4)}\right) + \mu\left(M_3^{(4)}\right)}, \tag{18}$$

with

$$\mu\left(M_1^{(4)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in C_{01}^{(4)}\}} dxdy$$

$$= \frac{\pi}{2} \text{ area } C_{01}^{(4)} = \frac{\pi}{2}\left(2ab - \frac{\pi m^2}{4}\right),$$

$$\mu\left(M_2^{(4)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in C_{02}^{(4)}\}} dxdy$$

$$= \frac{\pi}{2} \text{ area } C_{02}^{(4)} = \frac{\pi}{2}\left(4ab - \frac{\pi m^2}{4}\right),$$

$$\mu\left(M_3^{(4)}\right) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,\, y) \in C_{03}^{(4)}\}} dxdy$$

$$= \frac{\pi}{2} \text{ area } C_{03}^{(4)} = \frac{\pi}{2}\left(6ab - \frac{\pi m^2}{4}\right), \tag{19}$$

and considering the relations (17),

$$\mu\left(N_1^{(4)}\right)=\int_0^{\frac{\pi}{2}}d\varphi\iint_{\{(x,\,y)\in\widehat{C}_{01}^{(4)}(\varphi)\}}dxdy$$

$$=\int_0^{\frac{\pi}{2}}\left(\text{area }\widehat{C}_{01}^{(4)}(\varphi)\right)d\varphi$$

$$=\frac{\pi}{2}\left(2ab-\frac{m^2}{4}\right)$$

$$-(2a+b-m)\,l+\frac{l^2}{2},$$

$$\mu\left(N_2^{(4)}\right)=\frac{\pi}{2}\left(4ab-\frac{m^2}{4}\right)$$

$$-(4a+b-m)\,l+\frac{l^2}{2},$$

$$\mu\left(N_3^{(4)}\right)=\frac{\pi}{2}\left(6ab-\frac{m^2}{4}\right)$$

$$-(6a+b-m)\,l+\frac{l^2}{2}. \tag{20}$$

The relations (18), (19), and (20) give us

$$P_{\text{int}}^{(4)}=\frac{\dfrac{\pi(1-\pi)m^2}{4}+2(4a+b-m)l-l^2}{\pi\left(4ab-\dfrac{\pi m^2}{4}\right)}. \tag{21}$$

For $m=0$, the obstacles become points and the probability (21) becomes the probability P_1 of the previous section.

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